FLOW OF A VISCOUS FILM WITH FREE BOUNDARIES

IN THE ONE-DIMENSIONAL APPROXIMATION

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UDC 532.516

To describe the flow of a viscous film on the surface of a nonviscous liquid [1] proposed a system of equations (notation same as [1])

$$\frac{\partial H}{\partial t}+\frac{\partial H v_k}{\partial x_k}=0,\quad \rho H\frac{dv_i}{dt}=\frac{\partial S_{ik}}{\partial x_k},$$
$$S_{ik}=-p\delta_{ik}+3\mu H\frac{\partial v_j}{\partial x_j}\delta_{ik}+\mu H\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{\partial v_j}{\partial x_j}\delta_{ik}\right),$$
$$p=\rho\ (1-\rho/\rho_1)\ gH^2/2,\ i,\ j,\ k=1,\ 2. \tag{1}$$

In the expression for the stress tensor S_{ik} the first term is the net hydrostatic pressure over film thickness with consideration of the immersion of the film into the supporting liquid; the second term describes viscous forces related to film tension; and the third corresponds to shear deformation.

Let a film with free edges flow in the direction of the Ox axis, which we assume to be an axis of symmetry. We also assume that the film thickness H, longitudinal velocity u, and width W are functions of time and slowly varying functions of coordinate x. This allows one-dimensional description of the flow. On the lateral edges $y=\pm W/2$ let there act forces $F=(F_x/2,\ F_y)$, $F'=(F_x/2,\ -F_y)$, per unit edge length, symmetric about the x axis. Then the film is in tension in the transverse direction due to the forces F_y and $-F_y$, while in the longitudinal direction there acts on both edges a net force F_x. Applying to system (1) the same procedure by which it itself was derived from the three-dimensional Navier—Stokes equations, we obtain a system for H, W, u as functions of the variables x, t:

$$\partial HW/\partial t+\partial HWu/\partial x=0,$$
$$\rho HW\frac{du}{dt}=\frac{\partial}{\partial x}\left[W\left(-\frac{p}{2}+3\mu H\frac{\partial u}{\partial x}+\frac{F_y}{2}+\frac{F_x}{8}\frac{\partial W}{\partial x}\right)\right]+F_x,$$
$$\frac{dW}{dt}=\frac{W}{4\mu H}\left[F_y+p+\frac{F_x}{4}\frac{\partial W}{\partial x}\right]-\frac{W}{2}\frac{\partial u}{\partial x}.$$

In the momentum equation viscous effects are considered by the term $3\mu HW\partial u/\partial x$. As is well known, for extension flows the proportionality coefficient between viscous stresses and deformation rate is equal to 3μ [2]. In the mathematical description of such flows, which are found in problems of shaping various objects, it is usually assumed that no forces act on the lateral surface of the object and the configuration of its cross section does not change (see, for example, [3]). In the present case the changes in film thickness and width follow different laws.

The forces F_x, F_y may be found from system (2). The result may be interpreted as a solution of the identification problem, where the observed motion is used to determine the forces acting, or the control problem, where those forces are found, which must be applied to the film to obtain the desired flow regime.

For steady state flows the liquid volume flow rate will be constant: $Q=HWu$. Let $Q>0$, so that $u>0$. Then we obtain the following expressions for the forces:

$$F_x=\frac{d}{dx}\left(\rho Qu+Wp-4\mu HW\frac{du}{dx}-2\mu Hu\frac{dW}{dx}\right),$$
$$F_y=\frac{4\mu Hu}{W}\frac{dW}{dx}+2\mu H\frac{du}{dx}-p-\frac{F_x}{4}\frac{dW}{dx}.$$

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 67-70, July-August, 1983. Original article submitted June 16, 1982.

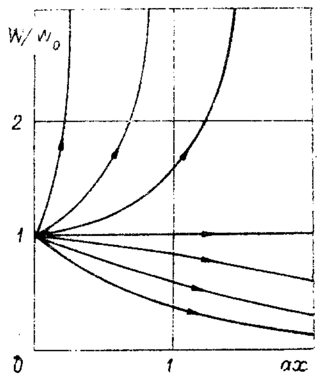

Fig. 1

For a film of constant width, if we neglect inertia and hydrostatic pressure, the expressions take on the simple form

$$F_x = -4Q\frac{d}{dx}\left(\mu\frac{d\ln u}{dx}\right),\quad F_y = \frac{2\mu Q}{W}\frac{d\ln u}{dx}.$$

Within the one-dimensional model we will consider the flow of a film when there acts orthogonally to its edge a force $-\gamma$, equal to the spreading coefficient [1], while $F_x = \gamma\partial W/\partial x$, $F_y = -\gamma$. Introducing the dimensionless film thickness $h = H/H_*$ (where H_* is the equilibrium thickness) and the constant $\varkappa = \rho g(1 - \rho/\rho_1)H_*/8\mu$, we rewrite system (2) in the form ($\nu = \mu/\rho$)

$$\frac{\partial hW}{\partial t} + \frac{\partial hWu}{\partial x} = 0,\quad \frac{dW}{dt} = W\left[\frac{\varkappa}{h}(h^2-1) - \frac{1}{2}\frac{\partial u}{\partial x}\right],$$
$$hW\frac{du}{dt} = \frac{\partial}{\partial x}\left\{W\nu\left[-2\varkappa(h^2-1) + 3h\frac{\partial u}{\partial x}\right]\right\}. \tag{3}$$

A system of equations practically equivalent to Eq. (3) for flows developing under the action of a longitudinal extensive force was solved numerically in [4], which presented a comparison of the calculation results with experimental data. For slow steady state flows Eq. (3) may be solved analytically. The last equation of the system has an integral

$$W\nu[-2\varkappa(h^2-1) + 3h\partial u/\partial x] = C.$$

If external extensive forces are absent, then $h \to 1$, $\partial u/\partial x \to 0$ as $x \to \infty$, hence $C = 0$. The remaining equations of system (3) take on the form

$$dh/dt + (4\varkappa/3)(h^2-1) = 0,\ dW/dt = (2\varkappa W/3h)(h^2-1).$$

At $x = 0$ let a film of width W_0 and thickness h_0 move with a velocity u_0. After a time interval $\tau = t - t_0$ the thickness of the film element with coordinate $x = 0$ at time t_0 will be

$$h(\tau) = 1 + 2/[e^{8\varkappa\tau/3}(h_0+1)/(h_0-1) - 1],$$

while its width and position are given by

$$W(\tau) = W_0(h_0/h(\tau))^{1/2},\quad x(\tau) = u_0\sqrt{h_0}\int_0^\tau d\tau/\sqrt{h(\tau)}.$$

As $x \to \infty$ the film width tends to the value $W_0\sqrt{h_0}$.

In considering effects produced by transverse extensive forces we will assume the film to be very viscous and neglect hydrostatic pressure and inertial forces. In the steady state at $F_x = 0$ from Eq. (2) we obtain

$$3\mu HW du/dx + WF_y/2 = C. \tag{4}$$

The constant C is defined by the boundary conditions. It is equal to the longitudinal force applied to the film section at the end $x = L$, if at that point $F_y = 0$. With consideration of the relationship $Q = HWu = H_0W_0u_0$, from Eq. (4) and the last equation of system (2) we obtain an equation for W:

$$dW/dx + CW/(6\mu Q) - F_y W^2/(3\mu Q) = 0. \quad (5)$$

For constant F_y and μ this equation has a solution (at $a = C/(6\mu Q)$, $b = 2F_y W_0/C$)

$$W = W_0/[b + (1 - b)e^{ax}]. \quad (6)$$

The remaining quantities are given by

$$u = u_0 \sqrt{b + (1 - b)e^{ax}}\, e^{3ax/2},\ H = H_0 \sqrt{b + (1 - b)e^{ax}}\, e^{-3ax/2}. \quad (7)$$

At $b = 1$ the film width remains constant, at $b < 1$ it decreases with increase in x, and at $b > 1$ it increases, tending at infinity to $x_* = (1/a)\ln[b/(b - 1)]$ (see Fig. 1). We will note two limiting cases. When transverse forces are small,

$$u = u_0 e^{2ax},\ H/H_0 = W/W_0 = \sqrt{u_0/u}. \quad (8)$$

When longitudinal forces are small,

$$W = W_0/(1 - F_y W_0 x/3\mu Q),\ u/u_0 = H/H_0 = \sqrt{W_0/W}. \quad (9)$$

Using Eq. (5), the action on the film of a transverse force $F_y = f\delta(x - x_0)$ can be determined. Denoting $W_\pm = W(x_0 \pm 0)$, for the change in width we have $1/W_+ = 1/W_- - f/3\mu Q$, while for the remaining quantities (in similar notation) $u_+/u_- = H_+/H_- = \sqrt{W_-/W_+}$.

We will consider the behavior of an elementary cylinder of length l of viscous material under the action of extensive forces F(t) applied to the end faces. From the equations

$$3\mu S\partial u/\partial x = F,\ \partial u/\partial x = (dl/dt)/l,\ lS = l_0 S_0$$

(where S is the cross-sectional area) it follows that $1/l = 1/l_0 - \int_0^t F dt/3\mu S_0 l_0$. The cylinder expands to infinity, $l = \infty$, when it receives a critical impulse $P_* = 3\mu S_0$, independent of its original length. It can be shown that this principle determines the position of the singular point for the solution of Eq. (9), as well as the maximum permissible value of transverse point force f.

Although the steady state flow of Eq. (8) exists formally at all x, the length of a material element becomes infinite when it receives an impulse equal to 3μ per unit cross-sectional area, i.e., in fact such a flow can exist only over a finite interval of the axis Ox. The situation is the same in the problem of a film falling under the action of its own weight, which in essence coincides with the problem of a viscous jet [5].

If the film expands simultaneously in two directions, then although the relationship $P_* = 3\mu S_0$ is no longer valid, the order of magnitude of the critical impulse which destroys the film does not change. Thus, for flows of the type of Eqs. (6), (7) its maximum value will not exceed $4\mu S_0$.

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